

# Fixed points of left reversible semigroup of isometry mappings in Banach spaces

S. Rajesh

Department of Mathematics

Indian Institute of Technology Tirupati, India - 517506

e-mail: srajeshiitmdt@gmail.com

## Abstract

In this paper, we prove the existence of a common fixed point in  $C(K)$ , the Chebyshev center of  $K$ , for a left reversible semigroup of isometry mappings. This existence result improves the results obtained by Lim et al. and Brodskii and Milman.

## 1 Introduction

Let  $K$  be a nonempty weakly compact convex set in a Banach space  $X$ . A map  $T : K \rightarrow X$  is said to be isometry if  $d(Tx, Ty) = d(x, y)$  for all  $x, y \in K$ . The notion of normal structure, which is introduced in [1], is defined as follows:

**Definition 1.1** [1, 5] *A nonempty bounded convex set  $K$  in a Banach space  $X$  is said to have normal structure if for every nonempty convex set  $C \subseteq K$  with  $\delta(C) > 0$  has a point  $x \in C$  such that  $r(x, C) < \delta(C)$ , where  $r(x, C) = \sup\{\|x - y\| : y \in C\}$  and  $\delta(C) = \sup\{\|z - y\| : z, y \in C\}$ .*

Define  $r(K) = \inf\{r(x, K) : x \in K\}$  and  $C(K) = \{x \in K : r(x, K) = r(K)\}$ . Then the set  $C(K)$  and the number  $r(K)$  are called, respectively, the set of Chebyshev center of  $K$  and the Chebyshev radius of  $K$ .

The notion of UKK-norm is defined as follows:

**Definition 1.2** [3, 5] *A Banach space  $X$  is said to have uniformly Kadec-Klee (UKK) norm if and only if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that*

$$\{x_n\} \subseteq B[0, 1], x_n \text{ converges weakly to } x_0, \text{ and} \\ \text{sep}\{x_n\} := \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon,$$

imply that

$$\|x_0\| \leq 1 - \delta.$$

**Definition 1.3** [2, 13] *A semigroup  $S$  is called left reversible if for every pair of elements  $a, b \in S$ , there exists a pair  $c, d \in S$  such that  $ac = bd$ .*

Now, we state the basic common fixed point theorem for left reversible semigroup of nonexpansive maps.

**Theorem 1.1** [13] *Let  $K$  be a nonempty compact convex set in a Banach space and  $\mathfrak{F}$  be a left reversible semigroup of nonexpansive self-maps on  $K$ . Then  $K$  contains a common fixed point of  $\mathfrak{F}$ .*

**Theorem 1.2** [10] *Let  $K$  be a nonempty weakly compact convex set in a Banach space  $X$  and assume that  $K$  has normal structure. Let  $S$  be a left reversible topological semigroup of nonexpansive, separately continuous actions on  $K$ . Then  $K$  contains a common fixed point for  $S$ .*

For the recent advancements in existence of common fixed points for various semigroups, one can refer to [7, 8, 9] and the reference therein.

The study of existence of fixed points in  $C(K)$  is initiated by Brodskii and Milman [1]. In fact, Brodskii and Milman proved:

**Theorem 1.3** [1] *Let  $K$  be a nonempty weakly compact convex subset of a Banach space  $X$  and  $\mathfrak{F} = \{T : K \rightarrow K : T \text{ is a surjective isometry mapping}\}$ . Assume that  $K$  has normal structure, then there exists  $x \in C(K)$  such that  $Tx = x$  for every  $T \in \mathfrak{F}$ .*

Motivated by this result (Theorem 1.3) of Brodskii and Milman and the fact  $T(C(K)) = C(K)$  whenever  $T$  is a surjective isometry on  $K$ , Lim et al. raised the following questions in [11]:

**Question 1.** Let  $T$  be an isometry on  $K$  which is not surjective. Does one still have  $T(C(K)) \subseteq C(K)$  ?

**Question 2.** Let  $K$  be a weakly compact convex subset in a Banach space  $X$  and assume that  $K$  has normal structure. Does there exist a point in  $C(K)$  which is fixed by every isometry from  $K$  into  $K$  ?

In case of uniformly convex Banach spaces, Lim et al. [11] affirmatively answered the above questions. Also, Lim et al. [11] established the next result:

**Theorem 1.4** [11] *Let  $K$  be a nonempty weakly compact convex set in a Banach space  $X$  and  $T$  be an isometry from  $K$  into  $K$ . Assume that  $K$  has normal structure. Then  $T$  has a fixed point in  $C(K)$ .*

In the setting of strictly convex Banach spaces, the authors in [14] proved:

**Theorem 1.5** [14] *Let  $K$  be a nonempty weakly compact convex set  $K$  having normal structure in a strictly convex Banach space  $X$  and  $\mathfrak{F}$  be a commuting family of isometry self-mappings on  $K$ . Then  $\mathfrak{F}$  has a common fixed point in the set of Chebyshev center,  $C(K)$ , of  $K$ .*

Moreover, the authors in [15] showed that the set of all Chebyshev center  $C(K)$  need not be invariant under isometry mappings.

In this paper, we prove that if  $K$  is a nonempty weakly compact convex set in a Banach space  $X$  and  $\mathfrak{F}$  is a left reversible semigroup of isometry mappings on  $K$  such that every sub-semigroup of  $\mathfrak{F}$  is also left reversible, then there exists a point  $x_0$  in  $C(K)$ , the set of Chebyshev center of  $K$ , such that  $Tx_0 = x_0$  for all  $T \in \mathfrak{F}$ , whenever either  $X$  has UKK-norm or  $X$  is strictly convex and  $K$  has normal structure.

## 2 Common fixed point theorems

Now, we state an interesting fact about isometry mappings on compact sets in metric spaces.

**Theorem 2.1** [6] *Let  $K$  be a nonempty compact set in a metric space and  $T : K \rightarrow K$  be an isometry map. Then  $T(K) = K$ .*

**Proof** Suppose  $T(K) \subsetneq K$ . Then there exists  $y \in K$  such that  $T^n(y) \notin T^{n+1}(K)$  for  $n = 0, 1, 2, \dots$ , where  $T^0(y) = y$ . Since  $T(K)$  is compact and  $y \notin T(K)$ , there exists  $z \in T(K)$  such that  $d(y, z) = \text{dist}(y, T(K))$ , where  $\text{dist}(y, T(K)) = \inf\{d(y, x) : x \in T(K)\}$ .

Also, it is easy to see that  $\text{dist}(T^n(y), T^{n+1}(K)) = \text{dist}(y, T(K))$ , for all  $n \in \mathbb{N}$ . Hence  $\text{dist}(T^n(y), T^{n+1}(K)) = d(y, z)$ , for all  $n \in \mathbb{N}$ .

Since  $\{T^n(y)\}$  is a sequence in the compact set  $K$ ,  $\{T^n(y)\}$  has a convergent subsequence, say  $\{T^{n_k}(y)\}$ . Suppose that  $\{T^{n_k}(y)\}$  converges to  $y_0 \in K$ . Note that  $T^n(K) \subseteq T^{n-1}(K)$ , for  $n \in \mathbb{N}$  and  $\{T^{n+1}(y)\} \subseteq T^n(K)$  for all  $n \in \mathbb{N}$ . Hence  $y_0 \in \bigcap_{n \in \mathbb{N}} T^n(K)$ .

As  $y_0 \in T^n(K)$  for  $n \in \mathbb{N}$ ,  $d(T^n(y), y_0) \geq \text{dist}(T^n(y), T^{n+1}(K)) = d(y, z)$ . Hence  $\lim_{k \rightarrow \infty} d(T^{n_k}(y), y_0) \geq d(y, z) > 0$ . But the sequence  $\{d(T^{n_k}(y), y_0)\}$  converges to  $d(y_0, y_0) = 0$ . This contradiction shows that  $T(K) = K$ .  $\square$

**Theorem 2.2** *Let  $K$  be a nonempty compact convex set in a Banach space  $X$  and  $\mathfrak{F}$  be a family of isometry self-maps on  $K$ . Then there exists  $x_0 \in C(K)$  such that  $Tx_0 = x_0$  for all  $T \in \mathfrak{F}$ .*

**Proof** Note that  $K$  has normal structure, as  $K$  is a compact convex set. Also, it follows from Theorem 2.1 that  $T(K) = K$  for all  $T \in \mathfrak{F}$ . Hence, by Theorem 1.3, there exists  $x_0 \in C(K)$  such that  $Tx_0 = x_0$ , for all  $T \in \mathfrak{F}$ .  $\square$

**Theorem 2.3** *Let  $K$  be a nonempty weakly compact convex set in a Banach space  $X$  with a UKK-norm. Let  $\mathfrak{F}$  be a left reversible topological semigroup of isometry, separately continuous actions on  $K$ . Assume that every sub-semigroup of  $\mathfrak{F}$  is also left reversible. Then the Chebyshev center of  $K$ ,  $C(K)$ , contains a common fixed point of  $\mathfrak{F}$ .*

**Proof** It follows from Theorem 1.4 that every map in  $\mathfrak{F}$  has a fixed point in  $C(K)$ . Let  $F_T = \{x \in C(K) : Tx = x\}$  for  $T \in \mathfrak{F}$ . It is known [3, 5] that if  $X$  has a UKK-norm, then  $C(K)$  is a nonempty compact subset of  $K$ . Hence it is enough to prove that the family  $\mathfrak{S} = \{F_T : T \in \mathfrak{F}\}$  has the finite intersection property.

Now, consider a finite subfamily  $\{T_1, T_2, \dots, T_m\}$  of  $\mathfrak{F}$ . Take  $m = 3$ , the same proof will go through for any  $m \geq 4$ . Let  $n \geq 3$  and  $\sigma \in \mathfrak{S}_n$ , where  $\mathfrak{S}_n$  is the permutation group.

Set  $T_{\sigma(j)} = T_l$  if  $\sigma(j) \equiv l \pmod{3}$  and  $T^\sigma = T_{\sigma(1)} \circ T_{\sigma(2)} \circ \dots \circ T_{\sigma(n)}$ . For  $n \geq m = 3$ , define  $W_n = \overline{\text{co}}(\bigcap_{\sigma \in \mathfrak{S}_n} T^\sigma(K))$ . As  $\mathfrak{F}$  is left reversible,  $W_n$  is nonempty for all  $n \geq 3$ . As for  $l = 1, 2, 3$ ,  $T_l(K) \subseteq K$ , we have  $W_{n+1} \subseteq W_n$  for  $n \geq 3$ .

Define  $K_0$  be the set of asymptotic center of  $\{W_n : n \in \mathbb{N}\}$  with respect to  $K$ , where  $W_1 = K = W_2$ .

It is claimed that  $T_l(K_0) \subseteq K_0$  for  $l = 1, 2, 3$ . Set  $V_{n+1,l} = \overline{co}(\bigcap_{\sigma \in \mathfrak{S}_n} T_l \circ T^\sigma(K))$ , for  $l = 1, 2, 3$ . Note that  $W_{n+1} \subseteq V_{n+1,l}$ , for  $n \geq 3$ .

Let  $x \in K_0$ . Then

$$\delta(T_l(x), W_{n+1}) \leq \delta(T_l(x), V_{n+1,l}) = \delta(x, W_n), \text{ for } n \geq 3.$$

Hence  $r(T_l(x)) = \inf \delta(T_l(x), W_n) \leq r(x)$ , for all  $x \in K_0$ . This implies that  $T_l(K_0) \subseteq K_0$ , for  $l = 1, 2, 3$  and consequently  $T(K_0) \subseteq K_0$  for all  $T$  in the semigroup  $\mathfrak{F}_0$  generated by  $\{T_1, T_2, T_3\}$ . Therefore, by Theorem 1.2, the semigroup  $\mathfrak{F}_0$  has a common fixed point in  $K_0$ . Let  $x_0 \in K_0$  be a common fixed point of  $\mathfrak{F}_0$ . Now note that for  $n \geq 3$

$$\delta(x_0, T^\sigma(K)) = \delta(x_0, K), \text{ for all } \sigma \in \mathfrak{S}_n.$$

Hence  $\delta(x_0, W_n) \leq \delta(x_0, K)$  for all  $n \in \mathbb{N}$ .

Also, note that as every sub-semigrupo of  $\mathfrak{F}$  is left reversible, there is a  $S_n \in \mathfrak{F}_0$  such that  $T^{\sigma_0} \circ S_n(K) \subseteq W_n$ , where  $\sigma_0$  is the identity permutation in  $\mathfrak{S}_n$ . Since  $x_0$  is a common fixed point of  $\mathfrak{F}_0$ , we have

$$\|x_0 - T^{\sigma_0} \circ S_n(y)\| = \|x_0 - y\|, \text{ for all } y \in K.$$

Therefore

$$\delta(x_0, K) = \delta(x_0, T^{\sigma_0} \circ S_n(K)) \leq \delta(x_0, W_n), \text{ for all } n.$$

Hence  $r(x_0) = \delta(x_0, K) \leq r(y) \leq \delta(y, K)$ , for all  $y \in K$ , as  $x_0 \in K_0$ . Consequently, the family  $\{T_1, T_2, T_3\}$  has a common fixed in  $C(K)$ . This proves that the family  $\mathfrak{S}$  has the finite intersection property. Therefore, the left reversible semigroup  $\mathfrak{F}$  has a common fixed point in  $C(K)$ .  $\square$

**Remark 2.1** *It is known [12, page-516] that a subgroup of an amenable group is amenable. Also, note that [4, 13] every left amenable semigroup is left reversible. Therefore, there are left reversible semigroups in which every sub-semigroup is also left reversible.*

**Theorem 2.4** *Let  $K$  be a nonempty weakly compact convex set having normal structure in a strictly convex Banach space  $X$ . Let  $\mathfrak{F}$  be a left reversible topological semigroup of isometry, separately continuous actions on  $K$ . Assume that every sub-semigroup of  $\mathfrak{F}$  is also left reversible. Then the Chebyshev center  $C(K)$  contains a common fixed point of  $\mathfrak{F}$ .*

**Proof** Note that  $F_T = \{x \in C(K) : Tx = x\}$  is a non-empty closed convex subset of  $C(K)$ , whenever  $T$  is an isometry on  $K$  as  $X$  is strictly convex. Hence it is enough to prove that the family  $\mathfrak{S} = \{F_T : T \in \mathfrak{F}\}$  has the finite intersection property.

Consider a finite subset  $\{T_1, T_2, \dots, T_m\}$  of  $\mathfrak{F}$ . Take  $m = 3$ . Define  $W_n$  and  $K_0$  as in Theorem 2.3. It is easy to see by the arguments in Theorem 2.3 that  $K_0$  is invariant under every member  $T$  in the semigroup  $\mathfrak{F}_0$  generated by  $\{T_1, T_2, T_3\}$ .

Now, by Theorem 1.2  $K_0$  contains a common fixed point of  $\mathfrak{F}_0$ . Let  $x_0$  be a common fixed point of  $\mathfrak{F}_0$ . Then  $x_0$  is a common fixed point of  $\{T_1, T_2, T_3\}$  and

$$\delta(x_0, T^\sigma(K)) = \delta(x_0, K), \text{ for all } \sigma \text{ in the permutation group } \mathfrak{S}_n.$$

Hence  $\delta(x_0, W_n) \leq \delta(x_0, K)$  for all  $n \geq 3$ . Now, by using the hypothesis that every sub-semigroup of  $\mathfrak{F}$  is also left reversible, it can be seen, by using arguments in Theorem 2.3, that  $r(x_0) = \inf \delta(x_0, W_n) = \delta(x_0, K)$ . But  $r(x_0) \leq r(y) \leq \delta(y, K)$ , for all  $y \in K$ , as  $x_0 \in K_0$ . This implies that  $x_0 \in C(K)$  and consequently the family  $\{T_1, T_2, T_3\}$  has a common fixed in  $C(K)$ . Therefore the family  $\mathfrak{S}$  has the finite intersection property. This shows that the left reversible semigroup  $\mathfrak{F}$  has a common fixed point in  $C(K)$ . □

## References

- [1] M. S. Brodskii, and D. P. Milman, *On the center of a convex set*, Dokl. Akad. Nauk SSSR **59** (1948), 837-840, in Russian.
- [2] Clifford, A. H. and G. B. Preston, *The algebraic theory of semigroups*, Volume 1. American Mathematical Society, Providence (1961).

- [3] Ayerbe Toledano, J. M.; Dominguez Benavides, T. and Lopez Acedo, G. *Measures of noncompactness in metric fixed point theory*. Operator Theory: Advances and Applications, 99. Birkhauser Verlag, Basel, 1997.
- [4] Granier, E., *A theorem on amenable semigroups*, Trans. Amer. Math. Soc. **111** (1964), 367–379.
- [5] K. Goebel, and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, Cambridge, 1990.
- [6] S. Kumaresan, *Topology of metric spaces*, Narosa Publishing House, New Delhi, 2005.
- [7] A.T.-M. Lau, *Fixed point property for reversible semigroup of non-expansive mappings on weak\*-compact convex sets*, in: Fixed Point Theory Appl., vol. 3, Nova Sci. Publ., Huntington, NY, 2002, pp. 167–172.
- [8] A.T.-M. Lau, Yong Zhang, *Fixed point properties of semigroups of non-expansive mappings*, J. Funct. Anal. **254** (2008), 2534–2554.
- [9] A.T.-M. Lau, *Normal Structure and Common Fixed Point Properties for Semigroups of Nonexpansive Mappings in Banach Spaces*, Fixed Point Theory Appl., (2010), 1–14.
- [10] T. C. Lim, *Characterization of Normal structure*, Proc. Amer. Math. Soc., **43** (1974), 313-319.
- [11] T. C. Lim, P. K. Lin, C. Petalas, and T. Vidalis, *Fixed points of isometries on weakly compact convex sets*, J. Math. Anal. Appl. **282** (2003), 1-7.
- [12] Day, M. M., *Amenable semigroup*, Illinois J. Math. **1** (1957), 509–544. [ 2
- [13] Mitchell, Theodore *Fixed points of reversible semigroups of nonexpansive mappings*. Kodai Math. Sem. Rep. **22** (1970) 322–323.
- [14] S. Rajesh and P. Veeramani, *Chebyshev Centers and Fixed Point Theorems*, J. Math. Anal. Appl., **422** (2015), no. 2, 880-885.

- [15] S. Rajesh and P. Veeramani, *Lim's Center and Fixed Point Theorems for Isometry Mappings*, preprint.